AN EXAMPLE CONCERNING BERGMAN COMPLETENESS

Włodzimierz Zwonek

Abstract. We construct a bounded plane domain which is Bergman complete but for which the Bergman kernel does not tend to infinity as the point approaches the boundary.

The disc with center at $a \in \mathbb{C}$ and radius $r > 0$ we denote by $\triangle(a,r)$. We denote also $E := \triangle(0,1)$. For $a \in \mathbb{C}$, $0 < r < R \leq \infty$ we denote the annulus $P(a,r,R) := \{z \in \mathbb{C} : r < |z-a| < R\}$.

Let D be a bounded domain in $\mathbb{C}^n$. Let us denote by $L_h^2(D)$ square integrable holomorphic functions on D. $L_h^2(D)$ is a Hilbert space with the scalar product induced from $L^2(D)$. Let us define the *Bergman kernel of* D

$$K_D(z) = \sup\{\frac{|f(z)|^2}{||f||^2_{L^2(D)}} : f \in L_h^2(D), f \not\equiv 0\}.$$

For the basic properties of the Bergman kernel and other functions introduced below see e.g [Jar-Pfl].

It is well-known that $\log K_D$ is a smooth plurisubharmonic function. Therefore, we may define

$$\beta_D(z;X) := \left(\sum_{j,k=1}^n \frac{\partial^2 \log K_D(z)}{\partial z_j \partial \bar{z}_k} X_j \bar{X}_k\right)^{1/2}, \ z \in D,\ X \in \mathbb{C}^n.$$

The function β_D is a pseudometric called *the Bergman pseudometric.*

For $w, z \in D$ we put

$$b_D(w,z) := \inf\{L_{\beta_D}(\alpha)\},$$

where the infimum is taken over piecewise C^1-curves $\alpha : [0,1] \mapsto D$ joining w and z and $L_{\beta_D}(\alpha) := \int_0^1 \beta_D(\alpha(t);\alpha'(t))dt$.

We call b_D *the Bergman distance of* D.

A bounded domain D is called *Bergman complete* if any b_D-Cauchy sequence is convergent to some point in D with respect to the standard topology of D.

Any bounded Bergman complete domain is pseudoconvex.

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The proof of the Bergman completeness is often based on the proof of the convergence of the Bergman kernel to infinity as the point approaches the boundary, i.e. the following property

$$(*) \qquad \lim_{D\ni z\to\partial D} K_D(z)=\infty.$$

All known Bergman complete domains have the property $(*)$. On the other hand there are domains satisfying $(*)$ which are not Bergman complete (take the Hartogs triangle). Let us recall some known results on Bergman completeness and the property $(*)$:

– if D is a bounded hyperconvex domain in $\mathbb{C}^n$, then D satisfies $(*)$ (see [Ohs 2]) and D is Bergman complete (see [Bło-Pfl] and [Her]),

– if D is a bounded domain in $\mathbb{C}$ satisfying $(*)$, then D is Bergman complete (see [Chen 2]),

– all other known examples of Bergman complete domains (i.e. non-hyperconvex) satisfy $(*)$, too (see [Chen 1], [Her], [Jar-Pfl-Zwo] and [Zwo]).

As already mentioned it has not been clear whether the condition $(*)$ is necessary for a domain to be Bergman complete. As we show below it is not the case. The example given by us is a bounded domain in $\mathbb{C}$ (Theorem 5). Let us underline here that the domain is given completely effectively. As a by-product we also get an effective example of a bounded fat domain in $\mathbb{C}$ not satisfying $(*)$ (see Corollary 3). For the non-effective proof of the existence of such a domain see [Jar-Pfl-Zwo].

Below we restrict our considerations only to one-dimensional domains.

For a domain $D\subset\mathbb{C}$ and a function $f\in\mathcal{O}(D)$ we denote $||f||_D:=||f||_{L^2_h(D)}$. For $f,g\in L^2_h(D)$ we denote $\langle f,g\rangle_D:=\int_D f\bar{g}d\lambda_2$.

For a fixed point $z_0\in\mathbb{C}$, $0<r<\infty$ we define $\mathcal{O}_0(P(z_0,r,\infty))$ as the set of holomorphic functions φ from $\mathcal{O}(P(z_0,r,\infty))$ such that their Laurent expansion in $P(z_0,r,\infty)$ is of the form $\varphi(z)=\sum_{n=1}^{\infty}\frac{a_{-n}}{(z-z_0)^n}$. For such a function we also denote $(\varphi)_{-1}(z):=\frac{a_{-1}}{z-z_0}$ and $(\varphi)_{-2}(z):=\sum_{n=2}^{\infty}\frac{a_{-n}}{(z-z_0)^n}$.

Let us formulate the following two simple estimates, which we shall use very extensively in the sequel:

Lemma 1. *Let $\varphi\in\mathcal{O}(\triangle(z_0,R))$ $(0<R<\infty)$. Then for any $0\le r\le R$ the following inequality holds:*

$$||\varphi||^2_{\triangle(z_0,r)}\le\frac{r^2}{R^2}||\varphi||^2_{\triangle(z_0,R)}.$$

Let $\varphi\in\mathcal{O}_0(P(z_0,r,\infty))$. Assume that $r\le s\le t$ and $r<t$. Then the following inequality holds:

$$||\varphi||^2_{P(z_0,s,t)}\le\frac{\log t-\log s}{\log t-\log r}||\varphi||^2_{P(z_0,r,t)}.$$

At this place let us write down some technical property that we shall use in the sequel. Namely, the function $u(x):=\frac{\log x-\log b}{\log x-\log a}$, $x>1$, where $0<a<b<1$, is increasing, so $u(x)\le u(2)$, $x\in(1,2)$. Moreover, $u(2)\le 2u(1)$, if a and b are small enough, for instance if $a,b\le\exp(-4)$.

Below we shall consider sequences of positive numbers $0<r_j<s_j<t_j$, $j=1,2,\ldots$ and points $z_1,z_2,\ldots\in E$ such that $\bar{\triangle}(z_j,t_j)\cap\bar{\triangle}(z_k,t_k)=\emptyset$ for any j,k

$1,2,\dots$, $j \neq k$, $0 \notin \bar{\triangle}(z_j, r_j)$, $j = 1,2,\dots$. Additionally, we assume that $z_N \to 0$. Then for such a fixed system of sequences we define domains

$$D_N := E \setminus (\bigcup_{j=N}^{\infty} \bar{\triangle}(z_j, r_j) \cup \{0\}),\ N = 1,2,\dots.$$

In the sequel we shall also denote $D := D_1$.

Additionally, we make some assumptions of the purely technical character that we impose on the sequences considered:
(1)

$$t_j < \exp(-4),\ r_j^2 < \frac{|z_j|^2}{2},\ \frac{r_j^2}{t_j^2}+\frac{s_j}{t_j}+\sqrt{\frac{2\log s_j}{\log r_j}} < 1,\ \frac{2\log t_j}{\log r_j}+\sqrt{\frac{2\log s_j}{\log r_j}}+\frac{s_j}{t_j} < 1,\ j = 1,2,\dots.$$

Our first aim is to find some sufficient conditions for the system of sequences considered above implying the following condition

$$\liminf_{D \ni z \to 0} K_D(z) < \infty. \tag{2}$$

Lemma 2. *Assume the following inequalities:*

$$\sum_{N=1}^{\infty} \frac{s_N}{t_N} < \infty,\ \sum_{N=1}^{\infty} \sqrt{\frac{\log s_N}{\log r_N}} < \infty, \tag{3}$$

$$\sum_{N=1}^{\infty} \frac{-1}{\log r_N} < \infty. \tag{4}$$

Then there is a positive constant C such that

$$K_D(z) \leq C(K_E(z) + \sum_{j=1}^{\infty}(\frac{1}{|z-z_j|^2(-\log r_j)} + \frac{r_j^2}{(|z-z_j|^2 - r_j^2)^2})),\ z \in D.$$

Corollary 3. *Let D be as above. Assume the convergence as in* (3) *and* (4)*. Assume also that $z_N > 0$, $N = 1,2,\dots$ and $\sum_{N=1}^{\infty}(\frac{-1}{z_N^2 \log r_N} + \frac{r_N^2}{(z_N^2 - r_N^2)^2}) < \infty$. Then* (2) *is satisfied.*

It is easy to see that having given a sequence $z_N \to 0$, $0 < z_N < 1$, $N = 1,2,\dots$, one may easily (completely effectively) construct sequence $\{r_N\}$ such that the assumptions from Corollary 3 are satisfied.

Proof of Corollary 3. In view of Lemma 2 for $-\frac{1}{2} < z < 0$ the following inequalities hold:

$$K_D(z) \leq C(K_E(-1/2) + \sum_{j=1}^{\infty}(\frac{-1}{\log r_j z_j^2} + \frac{r_j^2}{(z_j^2 - r_j^2)^2})).$$

The last expression is finite by the assumption of the Corollary. $\square$

Proof of Lemma 2. Fix for a while some $N > 0$. Consider arbitrary $F \in L_h^2(D_N)$. It is a simple consequence of the Laurent expansion of F in the annulus $P(z_N, r_N, t_N)$

that $F=f+g$ in D_N, where $f\in\mathcal{O}(D_{N+1})$ and $g\in\mathcal{O}_0(P(z_N,r_N,\infty))$. It is easy to see that $f\in L_h^2(D_{N+1})$ and $g\in L_h^2(P(z_N,r_N,R))$, where $1<R<\infty$. In view of Lemma 1 we have

$$||f||^2_{\triangle(z_N,r_N)}\leq\frac{r_N^2}{t_N^2}||f||^2_{\triangle(z_N,t_N)}\leq\frac{r_N^2}{t_N^2}||f||^2_{D_{N+1}}.$$

Consequently,

$$||f||^2_{D_N}=||f||^2_{D_{N+1}}-||f||^2_{\triangle(z_N,r_N)}\geq(1-\frac{r_N^2}{t_N^2})||f||^2_{D_{N+1}}. \tag{5}$$

On the other hand Lemma 1 gives the following estimates

$$\begin{aligned}&||g||^2_{D_N}\geq||g||^2_{P(z_N,r_N,t_N)}=||g||^2_{P(z_N,r_N,1+|z_N|)}-||g||^2_{P(z_N,t_N,1+|z_N|)}\geq\\ &(1-\frac{\log(1+|z_N|)-\log t_N}{\log(1+|z_N|)-\log r_N})||g||^2_{P(z_N,r_N,1+|z_N|)}\geq(1-2\frac{\log t_N}{\log r_N})||g||^2_{P(z_N,r_N,1+|z_N|)}.\end{aligned} \tag{6}$$

Now we want to find some upper estimates for the scalar product.

$$\begin{aligned}|\langle f,g\rangle_{D_N}|\leq|\langle f,g\rangle_{P(z_N,r_N,s_N)}|+|\langle f,g\rangle_{D_N\setminus\bar\triangle(z_N,s_N)}|\leq\\ ||f||_{P(z_N,r_N,s_N)}||g||_{P(z_N,r_N,s_N)}+||f||_{D_N\setminus\bar\triangle(z_N,s_N)}||g||_{D_N\setminus\bar\triangle(z_N,s_N)}.\end{aligned}$$

Since

$$||f||^2_{P(z_N,r_N,s_N)}\leq||f||^2_{\triangle(z_N,s_N)}\leq\frac{s_N^2}{t_N^2}||f||^2_{\triangle(z_N,t_N)}\leq\frac{s_N^2}{t_N^2}||f||^2_{D_{N+1}}$$

and

$$||g||^2_{D_N\setminus\bar\triangle(z_N,s_N)}\leq||g||^2_{P(z_N,s_N,1+|z_N|)}\leq\frac{\log(1+|z_N|)-\log s_N}{\log(1+|z_N|)-\log r_N}||g||^2_{P(z_N,r_N,1+|z_N|)},$$

the following inequality holds

$$\begin{aligned}&|\langle f,g\rangle_{D_N}|\leq\\ &\frac{s_N}{t_N}||f||_{D_{N+1}}||g||_{P(z_N,r_N,1+|z_N|)}+\sqrt{\frac{2\log s_N}{\log r_N}}||f||_{D_{N+1}}||g||_{P(z_N,r_N,1+|z_N|)}\leq\\ &\frac12(\frac{s_N}{t_N}+\sqrt{\frac{2\log s_N}{\log r_N}})(||f||^2_{D_{N+1}}+||g||^2_{P(z_N,r_N,1+|z_N|)}).\end{aligned} \tag{7}$$

Since

$$||F||^2_{D_N}=||f+g||^2_{D_N}=||f||^2_{D_N}+||g||^2_{D_N}+2\operatorname{Re}\langle f,g\rangle_{D_N},$$

the inequalities (5), (6) and (7) give the following estimates

$$\begin{aligned}||F||^2_{D_N}\geq||f||^2_{D_{N+1}}(1-\frac{r_N^2}{t_N^2}-\frac{s_N}{t_N}-\sqrt{\frac{2\log s_N}{\log r_N}})+\\ ||g||^2_{P(z_N,r_N,1+|z_N|)}(1-\frac{2\log t_N}{\log r_N}-\frac{s_N}{t_N}-\sqrt{\frac{2\log s_N}{\log r_N}}).\end{aligned} \tag{8}$$

More generally, using the Laurent expansion of $F \in L_h^2(D)$ in any annulus $P(z_j, r_j, t_j)$, $j = 1, \dots, N$ we may find $F_j \in \mathcal{O}_0(P(z_j, r_j, \infty))$ (the choice of this F_j is independent of N) and $F_0^N \in \mathcal{O}(D_{N+1})$ such that $F = F_0^N + F_1 + \dots + F_N$ on D_1, $F - F_j$ extends to a function holomorphic on $D \cup \bar{\triangle}(z_j, r_j)$. Note that $F_0^N \in L_h^2(D_{N+1})$ and $F_j \in L_h^2(P(z_j, r_j, R))$, $r_j < R < \infty$, $j = 1, \dots, N$.

Then in view of the inequality obtained in (8) applied recursively we get the following estimate

$$||F||_D^2 \geq$$

$$\sum_{k=1}^N (||F_k||^2_{P(z_k,r_k,1+|z_k|)}(1 - \frac{2\log t_k}{\log r_k} - \sqrt{\frac{2\log s_k}{\log r_k}} - \frac{s_k}{t_k}) \prod_{j=1}^{k-1}(1 - \frac{r_j^2}{t_j^2} - \frac{s_j}{t_j} - \sqrt{\frac{2\log s_j}{\log r_j}}))+$$

$$||F_0^N||^2_{D_{N+1}} \prod_{j=1}^{N}(1 - \frac{r_j^2}{t_j^2} - \frac{s_j}{t_j} - \sqrt{\frac{2\log s_j}{\log r_j}})).$$

The convergence of the series $\sum_{N=1}^\infty \frac{s_N}{t_N}$ implies the convergence of the series $\sum_{N=1}^\infty \frac{r_N^2}{t_N^2}$ and, consequently, (3) implies that the infinite product

$$\prod_{j=1}^{\infty}(1 - \frac{r_j^2}{t_j^2} - \frac{s_j}{t_j} - \sqrt{\frac{2\log s_j}{\log r_j}})$$

is positive.

Moreover, $\inf_{j=1,2,\dots}\{1 - \frac{2\log t_j}{\log r_j} - \sqrt{\frac{2\log s_j}{\log r_j}} - \frac{s_j}{t_j}\}$ is positive.

This altogether gives the existence of an $\varepsilon > 0$ such that for any N

$$||F||_D^2 \geq \varepsilon(||F_0^N||^2_{D_{N+1}} + \sum_{j=1}^N ||F_j||^2_{P(z_j,r_j,1+|z_j|)}). \tag{9}$$

Our next aim is to show the local convergence of F_0^N to a function F_0 holomorphic on $E_* = \bigcup_{N=1}^\infty D_N$. Then in view of (9) this convergence will imply that $F_0 \in L_h^2(E_*)$ (consequently, we may treat F_0 as an L_h^2-function on E). Note that the desired convergence follows from the local uniform convergence of the series $\sum_{j=k}^\infty |F_j(z)|$ on D_k for any $k = 1, 2, \dots$, which is proven below.

When we prove the above convergence then $F = F_0 + \sum_{j=1}^\infty F_j$ on D_1 and the following estimate will hold:

$$||F||_D^2 \geq \varepsilon(||F_0||_E^2 + \sum_{j=1}^\infty ||F_j||^2_{P(z_j,r_j,1+|z_j|)}). \tag{10}$$

Let us introduce some auxiliary functions:

$$\tilde{k}_{j,-1}(z) :=$$

$$\sup\{\frac{|(\varphi)_{-1}(z)|^2}{||\varphi_{-1}||^2_{P(z_j,r_j,1+|z_j|)}} : \varphi \in \mathcal{O}_0(P(z_j,r_j,\infty)),\ (\varphi)_{-1} \not\equiv 0\} =$$

$$\frac{1}{2\pi|z - z_j|^2(\log(1+|z_j|) - \log r_j)},$$

$$\tilde{k}_{j,-2}(z) :=$$

$$\sup\{\frac{|(\varphi)_{-2}(z)|^2}{||(\varphi)_{-2}||^2} : \varphi \in \mathcal{O}_0(P(z_j,r_j,\infty)),\ (\varphi)_{-2} \not\equiv 0\}$$

for any $j=1,2,\dots,$ $z\in P(z_j,r_j,1+|z_j|)$.

Simple computations of the L^2-norms (of the functions $(\varphi)_{-2}$ imply that there is some constant C (independent of j) such that

$$\tilde k_{j,-2}(z)\le CK_{P(z_j,r_j,\infty)}(z)=\frac{Cr_j^2}{|z-z_j|^4}K_E(\frac{r_j}{z-z_j})=\frac{Cr_j^2}{\pi(|z-z_j|^2-r_j^2)^2},\ z\in P(z_j,r_j,1+|z_j|),\ j=1,2,\dots.$$

Note that for any $k\le N$ the following inequalities hold

$$(\sum_{j=k}^N|F_j(z)|)^2\le(\sum_{j=k}^N(|(F_j)_{-1}(z)|+|(F_j)_{-2}(z)|))^2\le$$
$$(\sum_{j=k}^N||(F_j)_{-1}||_{P(z_j,r_j,1+|z_j|)}\tilde k_{j,-1}^{1/2}(z)+||(F_j)_{-2}||_{P(z_j,r_j,1+|z_j|)}\tilde k_{j,-2}^{1/2}(z)))^2\le$$
$$(\sum_{j=k}^N(\tilde k_{j,-1}(z)+\tilde k_{j,-2}(z)))(\sum_{j=k}^N(||(F_j)_{-1}||^2_{P(z_j,r_j,1+|z_j|)}+||(F_j)_{-2}||^2_{P(z_j,r_j,1+|z_j|)}))=$$
$$(\sum_{j=k}^N(\tilde k_{j,-1}(z)+\tilde k_{j,-2}(z)))\sum_{j=k}^N||F_j||^2_{P(z_j,r_j,1+|z_j|)},\ z\in D_k,$$

which finishes the proof of the desired properties of F_0 and (10) (use the estimates for $\tilde k_{j,-1},\tilde k_{j,-2}$, (9) and use the condition (4) to get for any k the local boundedness of the last expression, independently of N).

Now we may prove the required estimate. In view of (10) we get the following estimate

$$K_D(z)=\sup\{\frac{|F(z)|^2}{||F||_D^2}:F\in L_h^2(D),\ F\not\equiv 0\}\le\frac{1}{\varepsilon}\sup$$
$$\{\frac{(|F_0(z)|+\sum_{j=1}^\infty(|(F_j)_{-1}(z)|+|(F_j)_{-2}(z)|))^2}{||F_0||_E^2+\sum_{j=1}^\infty(||(F_j)_{-1}||^2_{P(z_j,r_j,1+|z_j|)}+||(F_j)_{-2}||^2_{P(z_j,r_j,1+|z_j|)})},\ F\not\equiv 0\},\ z\in D$$

(the functions F_j in the formula above come from the decomposition of F considered earlier). And then proceeding as earlier we have

$$K_D(z)\le\frac{1}{\varepsilon}\sup$$
$$\{\frac{\Big(||F_0||_EK_E^{1/2}(z)+\sum_{j=1}^\infty(||(F_j)_{-1}||_{P(z_j,r_j,1+|z_j|)}\tilde k_{j,-1}^{1/2}(z)+||(F_j)_{-2}||_{P(z_j,r_j,1+|z_j|)}\tilde k_{j,-2}^{1/2}(z))\Big)^2}{||F_0||_E^2+\sum_{j=1}^\infty(||(F_j)_{-1}||^2_{P(z_j,r_j,1+|z_j|)}+||(F_j)_{-2}||^2_{P(z_j,r_j,1+|z_j|)})}\}$$
$$\le\frac{1}{\varepsilon}(K_E(z)+\sum_{j=1}^\infty(\tilde k_{j,-1}(z)+\tilde k_{j,-2}(z))),\ z\in D,$$

which finishes the proof of the lemma (use the estimates for $\tilde k_{j,-1}$ and $\tilde k_{j,-2}$). $\square$

Remark 4. Note that the technical assumptions in (1) do not cause loss of generality for z from the neighbourhood of 0 (in particular, it does cause any loss of generality in Corollary 3). The convergence of the series in (3) and (4) implies that for j large enough the technical properties from (1) are always satisfied. Therefore, because of the localization principle of the Bergman kernel (see [Ohs 1]) the estimates as in Lemma 2 (for z from the neighbourhood of 0) remain valid without these technical assumptions.

Let us formulate our main result.

Theorem 5. *There is a bounded domain $D \subset \mathbb{C}$ such that $\liminf_{z \to \partial D} K_D(z) < \infty$ and D is Bergman complete.*

Proof. The domain stated in the theorem will be some of the domains considered earlier defined as D_1. Below we shall impose some conditions on the sequences implying that the domain has the property as desired. Certainly, the point at which the Bergman kernel will not tend to infinity will be 0 (all other points from the boundary force the Bergman kernel to diverge to infinity while tending to them).

Let us start with a sequence $x_n := \frac{1}{n^5}$, $n \geq 2$. We also define n^5 different points lying on the circle of radius x_n.

$$z_{n,j} := x_n \exp(i \frac{2j\pi}{n^5}),\ j = 0, \dots, n^5 - 1.$$

Note that there is some $C > 0$ such that $\frac{1}{Cn^{10}} \leq |z_{n,k} - z_{n,j}|$ and $|z_{n,0} - z_{n,1}| \leq \frac{C}{n^{10}}$ for any n and for any $j, k = 0, \dots, n^5 - 1$, $j \neq k$. Define

$$t_n := \frac{1}{3Cn^{10}},\ r_n := \exp(-n^{19}),\ s_n := \exp(-n).$$

We also define $y_n := \frac{x_n + x_{n+1}}{2}$.

Note that for any n $\bar{\triangle}(z_{n,j}, t_n) \cap \bar{\triangle}(z_{n,k}, t_n) = \emptyset$, $j \neq k$. We also easily see that for n, m large enough (for $n, m \geq n_0 \geq 2$) the circles $\partial \triangle(0, y_m)$ are disjoint from the discs $\bar{\triangle}(z_{n,j}, r_n)$, $j = 0, \dots, n^5 - 1$. Now we build a sequence $\{z_N\}$ by gluing together one by one the (finite) sequences $\{z_{n,j}\}_{j=0}^{n^5-1}$ (starting with $n = n_0$). We associate to them the sequences t_n, r_n and s_n in such a way that r_N (respectively, s_N, t_N), where N is such that z_N is associated to $z_{n,j}$, equals r_n (respectively, s_n, t_n). For indices large enough the sequences satisfy the technical assumptions from (1).

Note that the convergence as in (3) and (4) for the sequences just defined will be satisfied when we prove that

$$\sum_{n=n_0}^{\infty} n^5 a_n < \infty,$$

where a_n equals $\frac{s_n}{t_n}$ or $\sqrt{\frac{\log s_n}{\log r_n}}$ or $\frac{-1}{\log r_n}$. One can easily verify that this is the case.

One may also check that

$$\sup\{ \sum^{\infty} (\frac{n^5}{|y_m - x_n|^2(-\log r_n)} + \frac{n^5 r_n^2}{(|y_m - x_n|^2 - r_n^2)^2}) : m = n_0, n_0 + 1, \dots \} < \infty.$$

Therefore, applying Lemma 2, we easily see that there is some $M_1<\infty$ such that the following inequality holds

$$K_D(z)<M_1 \text{ for any } z\in\bigcup_{n=n_0}^{\infty}\partial\triangle(0,y_n)\subset D. \tag{11}$$

It follows from (11) that the Bergman kernel of D does not diverge to infinity as the point approaches 0.

On the other hand take any $n\geq n_0$ and take a point $z\in D\cap\partial\triangle(0,x_n)$.

Then

$$K_D(z)\geq\max\{\frac{\frac{1}{|z-z_{n,j}|^2}}{||\frac{1}{\cdot-z_{n,j}}||_D^2},\ j=0,\dots,n^5-1\}\geq\frac{\frac{n^{20}}{C^2}}{||\frac{1}{\cdot-x_n}||^2_{P(x_n,r_n,2)}}\geq \tag{12}$$
$$\frac{n^{20}}{2\pi C^2(\log 2-\log r_n)}=\frac{n^{20}}{2\pi C^2(n^{19}+\log 2)}\to_{n\to\infty}\infty.$$

Now we are ready to prove the Bergman completeness of D. Suppose that D is not Bergman complete. Then there is a Cauchy sequence $\{w_k\}$ with respect to the Bergman distance converging to the boundary (in the natural topology). It is easy to see that this sequence must converge to 0. Choosing if necessary a subsequence we get from the definition of the Bergman distance that there are a constant $M_2<\infty$ and a continuous function $\gamma:[0,1)\to D$ such that $\lim_{t\to0}\gamma(t)=0$, $\gamma_{|[0,1-\varepsilon]}$ is piecewise C^1 and $L_{\beta_D}(\gamma_{|[0,1-\varepsilon]})<M_2$ for any $\varepsilon\in(0,1)$. Note that the graph of γ must intersect any set $\partial\triangle(0,x_n)\cap D$ for $n\geq n_1$ with some $n_1\geq n_0$. Denote this point of intersection by v_n. Then it follows from the definition of the Bergman distance that the sequence $\{v_n\}$ is a Cauchy sequence with respect to the Bergman distance. But, additionally, it follows from (12) that $K_D(v_n)$ tends to infinity as n goes to infinity. We prove below that this is impossible, which will finish the proof. We follow the ideas from [Chen 2].

By a result from [Pfl] there is a function $f\in L_h^2(D)$ such that $||f||_D=1$ and $\frac{|f(v_{n_j})|^2}{K_D(v_{n_j})}\to1$ for some subsequence $\{z_{v_j}\}$ (see [Chen 1] or [Chen 2]). Since functions bounded near 0 are dense in $L_h^2(D)$ (see [Chen 2], Lemma 4), there exists a function $g\in L_h^2(D)$ such that $||f-g||_D\leq\frac{1}{2}$ and g is bounded near 0. Then we have

$$\frac{1}{2}\geq||f-g||_D\geq\frac{|f(v_{n_j})-g(v_{n_j})|}{\sqrt{K_D(v_{n_j})}}\geq\frac{|f(v_{n_j})|}{\sqrt{K_D(v_{n_j})}}-\frac{|g(v_{n_j})|}{\sqrt{K_D(v_{n_j})}}\to1$$

– contradiction. □

Remark 6. The last part of the proof of Theorem 5 is based on the density of functions from $L_h^2(D)$ locally bounded in 0 in the space $L_h^2(D)$. We quoted in this context the result from [Chen 2]. Actually, we may prove this result in the special case of domains considered in Lemma 2 directly with elementary methods (without the use of the solutions of the $\bar\partial$-problem). Namely, it follows from considerations in the proof of Lemma 2 that for any $F\in L_h^2(D)$ we have $F=F_0+\sum_{j=1}^{\infty}F_j$, where $F_j\in L_h^2(P(z_j,r_j,1+|z_j|))\cap\mathcal{O}_0(P(z_j,r_j,\infty))$, $j=1,2,\dots$ are as in the proof of Lemma 2, $F_0\in L_h^2(E)$ and the convergence of the series is locally uniform

on D. Define $G_N := F_0 + \sum_{j=1}^N F_j$. Then $G_N \to F$ locally uniformly. Assume that the convergence is in $L_h^2(D)$-norm. Then F_0 may be approximated in $L_h^2(E)$ by bounded holomorphic functions on E and the functions F_j may be approximated in $L_h^2(P(z_j, r_j, 1+|z_j|))$ by bounded holomorphic functions in $P(z_j, r_j, 1+|z_j|)$, $j = 1, 2, \ldots$. Consequently, this implies the density of $H^\infty(D)$ in $L_h^2(D)$ (so a little more than it follows from the result of Chen). To finish the proof it is sufficient to show that $\sum_{j=1}^N F_j$ tends to $\sum_{j=1}^\infty F_j$ in $L_h^2(D)$. But this can be seen from the considerations similar to that in the proof of Lemma 2. Namely take $1 \le k < l$. Then

$$||F_k+F_{k+1}+\ldots+F_l||_{D_k}^2 = ||F_k||_{D_k}^2+||F_{k+1}+\ldots+F_l||_{D_k}^2+2\operatorname{Re}\langle F_k, F_{k+1}+\ldots+F_l\rangle_{D_k}.$$

The last expression is not larger than (repeat the reasoning from the proof of (7))

$$(||F_k||_{P(z_k,r_k,1+|z_k|)}^2 + ||F_{k+1}+\ldots+F_l||_{D_{k+1}}^2)(1+\frac{s_k}{t_k}+\sqrt{\frac{2\log s_k}{\log r_k}}).$$

Repeating this reasoning we get that the last expression is not larger than

$$\sum_{j=k}^{l} ||F_j||_{P(z_j,r_j,1+|z_j|)}^2 \prod_{m=k}^{\min\{l-1,j\}} (1+\frac{s_m}{t_m}+\sqrt{\frac{2\log s_m}{\log r_m}}).$$

The assumptions on the convergence from (3) and (9) easily finish the proof.

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Instytut Matematyki, Uniwersytet Jagielloński, Reymonta 4, 30-059 Kraków, Poland

E-mail address: zwonek@im.uj.edu.pl